Some Open problems about Singular Spectral Measures

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Background

Figure: In memory of Ka-Sing Lau (1948-2021) and Jean-Pierre Gabardo (1958-2024)

Spectral sets

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Conjecture (Fuglede's conjecture,1974)

Ω is a spectral set if and only if Ω is a translational tile.

Spectral measures

Definition

Let μ be a Borel probability measure in $\mathbb{R}^d$ with compact support. We say that μ is a spectral measure if there exists a countable Λ such that $E(\Lambda) = \{e^{2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$.

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1. (Mutually orthogonal) $\widehat{\mu}(\lambda - \lambda') = 0$ for all $\lambda \neq \lambda' \in \Lambda$.
2. (Completeness or Parseval identity)

$$\sum_{\lambda\in\Lambda} \left| \int f(x)e_\lambda(x)d\mu(x)\right|^2 = \int |f|^2 d\mu(x), \forall f \in L^2(\mu).$$

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It is not hard to find absolutely continuous or purely discrete spectral measures.

Singular spectral measures

Fractal spectral measures

(**Jorgensen and Pedersen**) The first singular measures with exponential ONB:

Let μ_4 be the Cantor measure supported on the Cantor set of $1/4$-contractions.

$$\mu_4(E) = \frac{1}{2}\mu_4(4E) + \frac{1}{2}\mu_4(4E-2).$$

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$$\mu_4 = \left(\frac{\delta_0 + \delta_{2/4}}{2}\right) * \left(\frac{\delta_0 + \delta_{2/4^2}}{2}\right) * \left(\frac{\delta_0 + \delta_{2/4^3}}{2}\right) \cdots$$
$$= \nu_n * \nu_{>n}.$$

ν_n is the convolution of the first n discrete measure.

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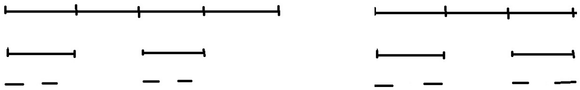
Theorem (Jorgensen and Pedersen, 1998)

μ_4 is a spectral measure with a spectrum

$$\Lambda = \left\{ \sum_{j=0}^{N-1} 4^j \epsilon_j : \epsilon \in \{0,1\} \right\}.$$



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Theorem (Jorgensen and Pedersen, 1998)

μ_3, the Cantor measures supported on Cantor sets of $1/3$ contractions,

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Still an open question!

Tiling equation

Proposition (Jorgensen and Pedersen, 1998)

μ is a spectral measure with a spectrum Λ if and only if

$$|\widehat{\mu}|^2 * \delta_\Lambda = |\sum_{\lambda\in\Lambda} |\widehat{\mu}(\xi-\lambda)|^2 = 1, \forall \xi \in \mathbb{R}^d.$$

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Fuglede's conjecture:

$$\exists \Lambda \text{ s.t. } |\widehat{\mathbf{1}}_\Omega|^2 * \delta_\Lambda = |\Omega|^2 \Longleftrightarrow \exists \mathcal{J} \text{ s.t. } \mathbf{1}_\Omega * \delta_{\mathcal{J}} = 1.$$

Connection to Tiling

Tiling for singular measures:
Let ν be the self-similar measure supported on $1/4$ Cantor set choosing digit $\{0,1\}$. Then

$$\mu * \nu = \mathcal{L}_{[0,1]}.$$

Hence, μ is also a translational tiling in the following sense.

$$\mu * (\nu * \delta_{\mathbb{Z}}) = m$$

where m is the standard Lebesgue measure on $\mathbb{R}^1$.

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2. (Dutkay and L., 2014) If μ is absolutely continuous, it must be a constant density.
3. If μ is purely discrete, e.g.

$$\mu = \sum_{r_n \in \mathbb{Q} \cap [0,1]} 2^{-n} \delta_{r_n}$$

If μ is spectral, it must be of equal weight and have finitely many points. (Łaba and Wang, 2006 or He, Lau, L., 2013 for

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3. (Hu and Lau, 2008 and Dai 2012) Let $\lambda = \frac{\sqrt{5}+1}{2}$. μ_λ is the Bernoulli convolution

$$\mu_\lambda(E) = \frac{1}{2}\mu_\lambda(\lambda E) + \frac{1}{2}\mu_\lambda(\lambda E - (1-\lambda)).$$

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4. They are just very special examples of the problem.

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Taking the distributional Fourier transform:

$$(\mu * \widetilde{\mu}) \cdot \widetilde{\delta_\Lambda} = \delta_0$$

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Spectral gap $a > 0$ for a tempered distribution T, is the largest $a > 0$ such that $\widehat{T}$ vanishes on a punctured open ball $B(0, a) \setminus \{0\}$.
Λ is translationally bounded if there exists $R > 0$ such that $\sup_{x \in \mathbb{R}^d} \#(\Lambda \cap B(x, R)) < \infty$.

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Suppose that Λ is translationally bounded and it has $D^+(\Lambda) = 0$. Then $\widehat{\delta_\Lambda}$ has a zero spectral gap. In particular, a (tight-frame) spectrum for a singular measure must have a zero spectral gap.

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Let μ be a singular measure. Suppose that **Support Conjecture** *holds and the support of $\mu * \widetilde{\mu}$ covers a neighborhood of the origin. Then μ cannot be (tight-frame) spectral.*

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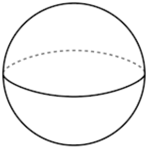
Theorem (L. Iosevich, Liu, Wyman, 2022)

Let K be a convex body on $\mathbb{R}^d$ with smooth boundary ∂K having everywhere positive Gaussian curvature and let σ be the surface measure supported on ∂K. Then the measure σ does not admit a Fourier frame.



Sphere vs polytope

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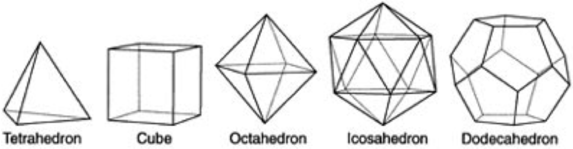
Let K be a polytope on $\mathbb{R}^d$ and let σ be the surface measure supported on ∂K. Then the measure σ is frame-spectral.



Additive measures

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Let μ and ν be two continuous Borel probability measures on $\mathbb{R}^1$. The additive space over μ and ν is the space $L^2(\rho)$, where ρ is the measure

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2. How about Riesz-spectrality or spectrality?

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Non-overlapping additive measure: $0 \notin (\text{supp}\mu) \cap (\text{supp}\nu)$.
Symmetric: $\mu = \nu$.

Theorem (Liu, Prince, L., 2021)

Let ρ be a non-overlapping symmetric additive measure with the component measure μ. Suppose that μ is Riesz-spectral. Then so is ρ.

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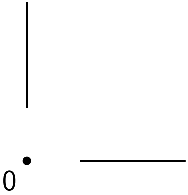

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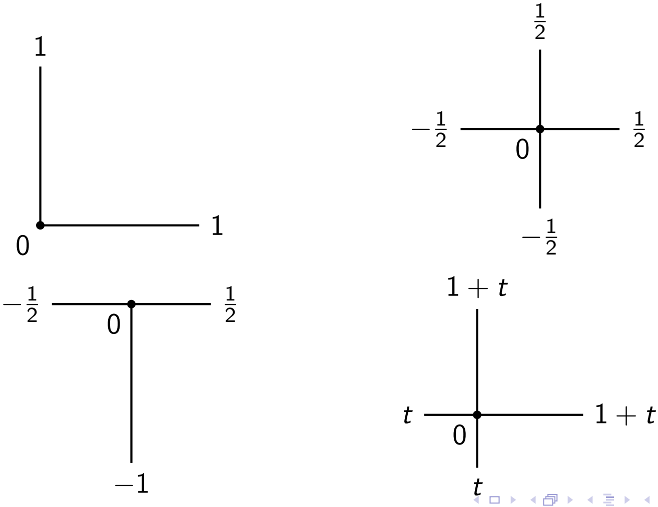

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2. *The T space and the* $t = -\frac{1}{2} + \frac{1}{2n}$*-Plus space does not admit any exponential orthonormal basis.*

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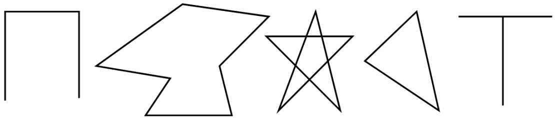
1. The whole problem was completely solved after some years of effort. (Ai-Lu-Zhou (2023), Kolountzakis-Wu (2025), Lu (2025))
2. (**Question:**) Can the boundary of polytopes be spectral?
3. By verifying the support conjecture, we solve the whole problem.

measures on line segments

Theorem (Kolountzakis and L., 2025)

1. *A finite union of line segments that forms a closed curve, self-intersecting or not, cannot be tight-frame spectral.*
2. *A finite union of line segments containing three lines that start at the same point and point in distinct directions cannot be tight-frame spectral.*



measures on line segments

Sketch of Proof (Square). Let μ be the boundary measure of the square and it is equal to

$$\mu = \frac{1}{4}(\mu_1 + \mu_2 + \mu_3 + \mu_4)$$

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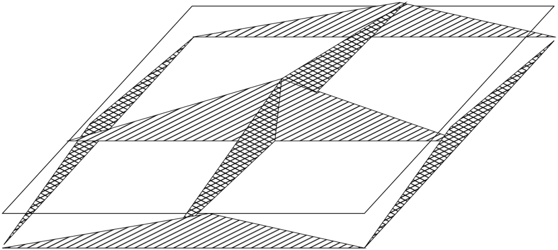
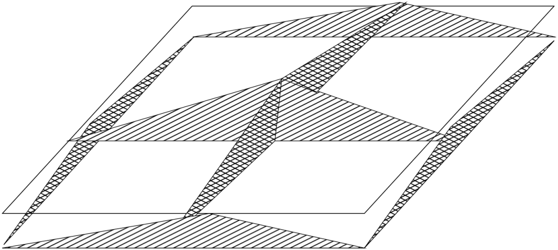
Parallel lines: A singular measure on the line segment.

measures on line segments



Main Claim: If Λ is a spectrum for μ, then $\widehat{\delta_\Lambda}$ has no support in the above square.

measures on line segments



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The claim implies there is a spectral gap for δ_Λ which is a contradiction since Λ is a spectrum for a singular measure.

Smooth part

Theorem (Support Conjecture holds for smooth $\mu * \widetilde{\mu}$)

Suppose (μ, Λ) is a tight-frame spectral pair such that $\mu * \widetilde{\mu}$ is absolutely continuous in the open set $U \not\ni 0$ and has a smooth, strictly positive density therein. Then supp$(\widehat{\delta_\Lambda}) \cap U = \varnothing$

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But h is positive, for all smooth φ supported on U,

$$\delta_\Lambda(\varphi) = h\widehat{\delta_\Lambda}(\varphi/h) = 0.$$

measures on line segments

Theorem (classic tempered distribution theorem)

Suppose that T *a tempered distribution supported on* $\mathbb{R}\times\{0\}$*. Then*

$$\langle T,\varphi\rangle=\sum_{j=0}^{J}\left\langle T_j,\frac{\partial^j}{\partial^j x_2}\varphi|_{x_2=0}\right\rangle$$

for some tempered distribution T_j *on* $\mathbb{R}^1$*.*

Final proof

Lemma (Key Lemma)

Suppose $F \in L^{\infty}(\mathbb{R}^2)$ *and* $T = \widehat{F}$*, a tempered distribution, has* *supp*$(T) \subset \mathbb{R} \times \backslash\{0\}$. *Then*
(a) there exists a distribution T_1 *on* $\mathbb{R}$ *such that for any* $h \in \mathcal{S}(\mathbb{R}^2)$ *we have*

$$T(h) = T_1(h(\cdot, 0)), \text{ and}$$

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4. Fubini's and projection of the spectrum is still a tight frame, we see that the integral is finite, a contradiction.

Riesz bases of exponentials

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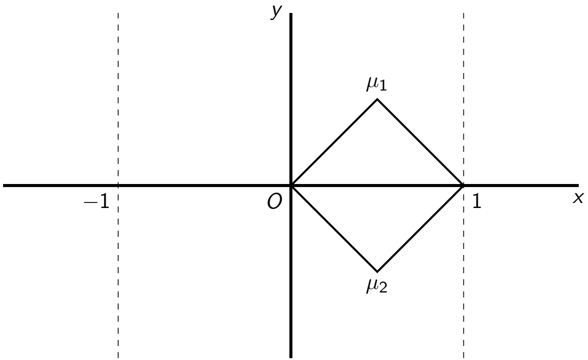
7. Bounded Multi-tiles by full-rank lattices admit RB (Kolountzakis, 2015), (Lev and Grepstad, 2014).

Riesz bases of exponentials

For a square boundary, it can be regarded as a multi-tiling by a closed subgroup $\mathbb{Z} \times \mathbb{R}$.



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Unfortunately, it does not admit Riesz basis in the form of finite union of lattices induced by the multi-tiling.

Theorem (L. and Sheynis, 2023)

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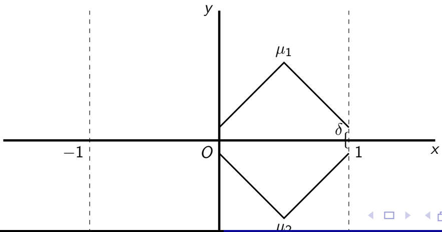
However, the following does.



Thank you